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Antoine Douai *
Laboratoire J.A Dieudonné, UMR CNRS 7351,
Université de Nice, Parc Valrose, F-06108 Nice Cedex 2, France
Email address: Antoine.DOUAI@unice.fr

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Abstract

We study Gauss-Manin systems of non tame Laurent polynomial functions. We focuse on Hori-Vafa models, which are the expected mirror partners of the small quantum cohomology of smooth hypersurfaces in weighted projectives spaces.

1 Introduction

A point in mirror symmetry is that it suggests the study of new, and sometimes unexpected, phenomena, on the A-side (quantum cohomology) as well on the B-side (singularities of regular functions). From this point of view, the case of (the contribution of the ambient part to) the small quantum cohomology of smooth hypersurfaces in weighted projective spaces is particularly significant and leads to the study of a remarkable class of regular functions on the torus, the Hori-Vafa models, see [13], [16], [22] and section 5. The key point is that, unlike the usual absolute situation, see for instance [9], [10], [11], [12], [23], such functions are not tame and may have some singular points at infinity (recall in few words that f is tame if the set outside which f is a locally trivial fibration is made from critical values of f and that these critical values belong to this set only because of the critical points at finite distance, see section 2.1). In this way, a geometric situation requires wild functions and this is the opportunity to study them more in detail. One aim of these notes is to enlighten this interaction between *singularities of functions (including at infinity)*, *Gauss-Manin systems*, *smooth hypersurfaces in weighted projective spaces*, *quantum cohomology* and to connect rather classical results in various domains. For instance, it's worth to note that an arithmetical condition that ensures the smoothness of a hypersurface in a weighted projective space gives also a number of vanishing cycles at infinity for the expected mirror partner, see sections 5.3 and 6.1.

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We proceed as follows: in a first part, we focuse on Gauss-Manin systems of (possibly wild) regular functions and their Brieskorn modules, emphasizing their relations with singular points (including at infinity), bifurcation set *etc...*, see sections 2 and 3. It happens that (and this is a major difference with the tame case) the Brieskorn module of a Hori-Vafa model f is not of finite type because the rank of the (localized Fourier transform of the) Gauss-Manin system G of f (see sections 3 and 5 for the definition of G) is strictly greater than the number of critical points at finite distance. The difference between the rank of G and the number of critical points at finite distance should be seen as a number of vanishing cycles at infinity. We discuss an explicit characterization of these singular points at infinity and their contribution to the Gauss-Manin system of f. Notice that the situation is slightly different from the classical polynomial case considered in [1], [21] *etc...*: as a Hori-Vafa model is a Laurent polynomial we have also to take into account the singular points on the polar locus of f at finite distance. Fortunately, the results in [28], [29] fit in very well with this situation.

In a second part, we are interested in the following formulation of mirror symmetry: above the small quatum cohomology of a degree d hypersurface in a projective space (and we consider here only the contribution of the ambient space to the small quantum cohomology, see [3], [13], [19] and section 7.2.1) and above a Hori-Vafa model on the B-side, we make grow a quantum differential system in the sense of [9], [10]. Two models will be mirror partners if their respective quantum differential systems are isomorphic. On the B-side, the expected quantum differential system can be constructed solving a Birkhoff problem for the Hori-Vafa model alluded to, as in the absolute case (*i.e* $d = 0$, see for instance [9], [10], [11], [12], [23]...), see section 6. In the tame case, this bundle is provided by the Brieskorn module as defined in section 3.3, which is in this situation a lattice in G: a difficult point of the theory is to verify that the Brieskorn module is indeed free of finite rank, and this follows from the tameness assumption. But, and as previously noticed, it will be certainly not the case for Hori-Vafa models, and we have to imagine something else. We give a general result in this way for quadrics in $\mathbb{P}^n$, and this was, after [14], one of the triggering factors of this paper. Precisely, let G be the (localized Fourier transform of the) Gauss-Manin system of the Hori-Vafa model of a smooth quadric in $\mathbb{P}^n$, see sections 3 and 5. We show in section 6.2 the following result:

Theorem 1.0.1 *We have a direct sum decomposition*

$$G = H \oplus H^{\circ}$$

of free modules, H being free of rank n and being equipped with a connection making it isomorphic to the differential system associated with the small quantum cohomology of quadrics in $\mathbb{P}^n$.

It follows that the rank of G is greater or equal than n and that it is equal to n if and only if $H^{\circ} = 0$. This is what happens for instance for $n = 3$ et $n = 4$, see example 5.3.1, and this is what it is expected in general, see conjecture 5.3.4. The case $n = 4$ is also considered in [14], using a different strategy.

These notes are organized as follows: in section 2 we discuss about tameness of regular functions and we study their Gauss-Manin systems in section 3. In section 4 we gather the results about hypersurfaces in weighted projective spaces that we need in order to define Hori-Vafa models in section 5. Their relationship with mirror symmetry is emphasized in section 6. As an application, we study the case of the quadrics in section 6.2.

2 Topology and tameness of regular functions

We collect in this section the general results about topology of regular functions that we will need. Our references are [5], [6] and [23]. The exposition is borrowed from the old preprint [8].

2.1 Isolated singularities including at infinity

Let U be an affine manifold of dimension $n \geq 2$, $S = \mathbb{C}$ and $f : U \to S$ be a regular function. We will say that f has *isolated singularities including at infinity* if there exists a compactification

$$\overline{f} : X \to S$$

of f, X is quasi-projective and $\overline{f}$ is proper, such that the support Σ_s of $\varphi_{\overline{f}-s} Rj_* \mathbb{C}_U$ is at most a finite number of points. Here, $j : U \to X$ denotes the inclusion and φ denotes the vanishing cycles functor [5, Chapter 4]. If it happens to be the case, f has at most isolated critical points on U [5, Theorem 6.3.17]. If moreover $\Sigma_s \subset U$ for all $s \in S$, f is said to be *cohomologically tame* [23].

Let us assume that f has isolated singularities including at infinity. Since ${}^p\varphi := \varphi[1]$ preserves peverse sheaves, $\mathcal{E}_s := {}^p\varphi_{\overline{f}-s} Rj_* \mathbb{C}_U[n]$ is a perverse sheaf with support in Σ_s and thus $\mathcal{H}^i(\mathcal{E}_s) = 0$ for $i \neq 0$, because Σ_s has ponctual support, see [5, Example 5.2.23]. For $x \in \Sigma_s$, the fibre $E_x := \mathcal{H}^0(\mathcal{E}_s)_x$ is a finite dimensional vector space. More precisely,

- if $x \in U$ we have
$$\dim E_x = \mu_x \text{ and } \sum_{x \in U} \dim E_x = \mu \tag{1}$$
μ_x denoting the Milnor number of f at x and μ the global Milnor number of f, see [5, proposition 6.2.19],
- if $x \in \Sigma_s \cap (X - U)$ we define
$$\nu_{x,s} := \dim E_x,\ \nu_s := \sum_{x \in \Sigma_s \cap (X-U)} \nu_{x,s} \text{ and } \nu := \sum_{x \in X-U} \nu_{x,s}. \tag{2}$$

In particular f is cohomologically tame if and only if $\nu = 0$.

Definition 2.1.1 *Assume that f has isolated singularities including at infinity. The point $x \in \Sigma_s \cap (X - U)$ is a singular point of f at infinity if $\nu_{x,s} > 0$.*

Let ${}^p\mathcal{H}^i$ be the perverse cohomology functor : one has, see for instance [5, Theorem 5.3.3]

$$DR(\mathcal{M}^{(i)}) = {}^p\mathcal{H}^i(Rf_* \mathbb{C}_U[n])$$

If f has isolated singularities including at infinity, the perverse sheaves ${}^p\mathcal{H}^i(Rf_*\mathbb{C}_U[n])$ are locally constant on S for $i \neq n$ because $\varphi_{t-s}({}^p\mathcal{H}^i(Rf_*\mathbb{C}_U[n])) = 0$ if $i \neq n$ for all $s \in \mathbb{C}$ [6, 3.1.1] and [5, Exercise 4.2.13]. It follows that $\mathcal{H}^0({}^p\mathcal{H}^i(Rf_*\mathbb{C}_U[n])) = 0$ and that $\mathcal{H}^{-1}({}^p\mathcal{H}^i(Rf_*\mathbb{C}_U[n]))$ is a constant sheaf on S for $i \neq n$. One has also, using the characterization of perverse sheaves in dimension 1 [5, Proposition 5.3.6],

$$0 \to \mathcal{H}^0({}^p\mathcal{H}^i(Rf_*\mathbb{C}_U)) \to R^i f_* \mathbb{C}_U \to \mathcal{H}^{-1}({}^p\mathcal{H}^{i+1}(Rf_*\mathbb{C}_U)) \to 0 \tag{3}$$

and therefore

$$ {}^p\mathcal{H}^i(Rf_*\mathbb{C}_U) = (R^{i-1}f_*\mathbb{C}_U)[1] \tag{4} $$

for all $i < n$ because $\mathcal{H}^0({}^p\mathcal{H}^i(Rf_*\mathbb{C}_U)) = 0$ for $i < n$. Notice also that

$$ {}^p\mathcal{H}^n(Rf_*\mathbb{C}_U) = (R^{n-1}f_*\mathbb{C}_U)[1] \tag{5} $$

if $R^n f_*\mathbb{C}_U = 0$.

We will use the next proposition it in order to compute the rank of the Fourier transform of the Gauss-Manin system of some regular functions, see theorem 3.1.2.

Proposition 2.1.2 ([5], [6]) *Let $f : U \to S$ be a regular function, with isolated singularities including at infinity.*

1. *One has*
$$ m = \mu + \nu + h^{n-1}(U) - h^n(U) \tag{6} $$
where m is the rank of $R^{n-1}f_*\mathbb{C}_{U|V}$, $V = S - \Delta$ denoting the maximal open set in S on which the restriction of $R^{n-1}f_*\mathbb{C}_U$ is a local system.

2. *One has*
$$ \chi(f^{-1}(s')) - \chi(f^{-1}(s)) = (-1)^{n-1}(\mu_s + \nu_s) \tag{7} $$
for all $s, s' \in S$ such that $s' \notin \Delta$.

Proof. 1. We give the proof in order to test the definitions. We have, for $\mathcal{F}^\bullet \in D^b_c(S)$,

$$ \chi(S, \mathcal{F}^\bullet) = \chi(S)\chi(S, \mathcal{F}^\bullet_x) - \sum_{s\in S} \chi(\varphi_{t_s}\mathcal{F}^\bullet) $$

where $x \in S$ is a generic point and $\chi(S, \mathcal{F}^\bullet) = \sum(-1)^{p+q}\dim H^p(S, \mathcal{H}^q(\mathcal{F}^\bullet))$, see [5, Exercise 4.2.15]. Applying this formula to $\mathcal{F}^\bullet = \mathcal{P} = {}^p\mathcal{H}^n(Rf_*\mathbb{C}_U)$, we get

$$ m + \chi(S, \mathcal{P}) = \sum_{s\in\Delta} \dim\, {}^p\varphi_{t_s}\mathcal{P} = \mu + \nu $$

because f has isolated singularities including at infinity. Because f is affine, we have $R^n f_*\mathbb{C}_U = 0$ and thus, using the exact sequence (3), $\mathcal{H}^0(\mathcal{P}) = 0$. Finally,

$$ \chi(S, \mathcal{P}) = \dim H^1(S, \mathcal{H}^{-1}\mathcal{P}) - \dim H^0(S, \mathcal{H}^{-1}\mathcal{P}) $$

where $\mathcal{H}^{-1}\mathcal{P} = R^{n-1}f_*\mathbb{C}_U$ by (5). We have also (Leray)

$$ 0 \to H^1(S, R^{i-1}f_*\mathbb{C}_U) \to H^i(U, \mathbb{C}) \to H^0(S, R^i f_*\mathbb{C}_U) \to 0 $$

for all i : if $i = n$, and because f is affine, we get $H^1(S, R^{n-1}f_*\mathbb{C}_U) = H^n(U)$; if $i = n-1$ we get $H^0(\mathcal{H}^{-1}\mathcal{P}) = H^{n-1}(U)$ because $R^{n-2}f_*\mathbb{C}_U$ is a constant sheaf on S. This gives the expected equation (6).
2. Analogous proof, see [5, Proposition 6.2.19]. □

Remark 2.1.3 *1. Formula (7) shows that the number of vanishing cycles at infinity ν defined by (2) is precisely the one defined by Siersma and Tibar and denoted by λ in [28, corollary 4.10], [29, paragraphe 3]. It also shows that singular points at infinity give a contribution to the bifurcation set of f, see section 2.3.*
2. Formula (7) has also another important consequence: if f has isolated singularities including at infinity, the numbers ν_s and ν do not depend on the choosen compactification of f.

2.2 A particular case : vanishing cycles at infinity with respect to the projective compactification by the graph

We apply the previous definitions to Laurent polynomials, using the standard compactification by the graph. We follow here [28] and [29].

Let $Y = \mathbb{P}^n$ and

$$F : Y \dashrightarrow \mathbb{P}^1$$

be the rational function defined by $F(x) = (P(x) : Q(x))$ where P and Q are two homogeneous polynomials of same degree. Let

$$G = \{(x,t) \in (Y - A) \times \mathbb{P}^1 \mid F(x) = t\}$$

where $A = \{x \in Y | \; P(x) = Q(x) = 0\}$ and

$$\mathbb{Y} = \{(x, (s:r)) \in Y \times \mathbb{P}^1 \mid rP(x) = sQ(x)\} \tag{8}$$

be the closure of G in $Y \times \mathbb{P}^1$. By definition, G is the graph of F restricted to $Y - A$ and thus $G \simeq Y - A$. Finally, the inclusion $Y - A \hookrightarrow \mathbb{Y}$ defines the compactification

$$\begin{array}{ccc} Y - A & \hookrightarrow & \mathbb{Y} \\ & \searrow & \downarrow \pi \\ & & \mathbb{P}^1 \end{array}$$

of F, π denoting the projection on the second factor. With the notations of section 2.1, $X = \mathbb{Y}$ and $\pi = \overline{f}$. The singular locus $\mathbb{Y}_{sing}$ of $\mathbb{Y}$ is contained in A.

Assume now that the hypersurface $\mathbb{Y}_a := \pi^{-1}(a)$ has an *isolated* singularity at $(p,a) \in A \times \{a\}$ and denote by $\mu_{p,a}$ the corresponding Milnor number. If $\mathbb{Y}_{sing}$ is a curve at (p,a), it intersects $\mathbb{Y}_s$, s close to a, at points $p_i(s)$, $1 \leq i \leq k$. Let $\mu_{p_i(s),s}$ be the Milnor number of $\mathbb{Y}_s$ at $p_i(s)$.

Proposition 2.2.1 *Assume that $\mathbb{Y}_a$ has an isolated singularity at $(p,a) \in A \times \{a\}$. Then*

$$\nu_{p,a} = \mu_{p,a} - \sum_{i=1}^{k} \mu_{p_i(s),s}.$$

Proof. Follows from remark 2.1.3 (1) and [28, Theorem 5.1]. □

This proposition is very explicit when $\mathbb{Y}_{sing}$ is a line $\{p\} \times \mathbb{C}$ (or a union of lines): indeed, let $\mu_{p,gen}$ be the Milnor number of the hypersurface $\mathbb{Y}_s$ at p for generic s.

Corollary 2.2.2 *Assume that $\mathbb{Y}_{sing} = \{p\} \times \mathbb{C}$. Then $\nu_{p,a} = \mu_{p,a} - \mu_{p,gen}$.* □

Remark 2.2.3 *We will apply the previous construction to Laurent polynomials*

$$f(x_1, \cdots, x_n) = \frac{P(x_1, \cdots, x_n)}{Q(x_1, \cdots, x_n)}$$

where P and Q have no common factors, Q is monomial and $\deg P \geq \deg Q$. *The homogeneization*[1] *of f is*

$$\frac{P(X_0, X_1, \cdots, X_n)}{Q(X_0, X_1, \cdots, X_n)} := \frac{X_0^{\deg P} P(X_1/X_0, \cdots, X_n/X_0)}{X_0^{\deg P} Q(X_1/X_0, \cdots, X_n/X_0)}$$

and we will write, for $t \in \mathbb{C}$,

$$F(X_0, X_1, \cdots, X_n, t) := P(X_0, X_1, \cdots, X_n) - tQ(X_0, X_1, \cdots, X_n)$$

We will have to distinguish two kinds of singular points at infinity: the ones on the hyperplane at infinity $X_0 = 0$ and the ones on the polar locus at finite distance. In the former case, $p = (0 : 1 : a_2 : \cdots : a_n)$ while in the latter case $p = (1 : a_1 : \cdots : a_n)$ with $a_1 \cdots a_n = 0$.

Remark 2.2.4 *How to recognize functions that do not have singular points at infinity?*
1. The polynomial case. *By [21, Theorem 1.3], a polynomial function f is cohomologically tame for the standard projective compactification by the graph if and only if f satisfies Malgrange's condition*

$$\exists \delta > 0, \ |x||\partial f(x)| \geq \delta \text{ for } |x| \text{ large enough,}$$

$\partial f(x)$ denoting the gradient of f at x. One can strengthen this condition and use Broughton's condition [1]: let us define (and we use here the standard compactification)

$$T_\infty(f) = \{c \in \mathbb{C} | \ \exists \ (p_n), \ p_n \to p \in X - U, \ \operatorname{grad} f(p_n) \to 0, \ f(p_n) \to c\} \tag{9}$$

Then f is cohomologically tame if $T_\infty(f) = \emptyset$.

2. Laurent polynomial case. *One can also write Malgrange's condition and Broughton's conditions as in [30, 1.3] using formula (8) but one has also to take into account the points $p \in X - U$ on the polar locus at finite distance and for which the previous conditions should be slightly different. The point is that one can have $T_\infty(f) = \emptyset$, where $T_\infty(f)$ is defined by (9), for a non cohomologically tame Laurent polynomial function f, see example 5.3.2 and example 5.3.1 below. This leads to the following definitions. Let f be a Laurent polynomial:*

- *if p is a point on the polar locus at finite distance, we define*

$$T_\infty^{fin}(f) = \{c \in \mathbb{C} | \ \exists \ (p_n), \ p_n \to p, \ p_n \operatorname{grad} f(p_n) \to 0, \quad f(p_n) \to c\} \tag{10}$$

- *if p is a point on the hyperplane at infinity, we define*

$$T_\infty^\infty(f) = \{c \in \mathbb{C} | \ \exists \ (p_n), \ p_n \to p, \ \operatorname{grad} f(p_n) \to 0, \ f(p_n) \to c\} \tag{11}$$

The expected result is that f is cohomologically tame if $T_\infty^{fin}(f) = T_\infty^\infty(f) = \emptyset$.

[1] If $n = 2$, we will denote by X, Y, Z the homogeneous coordinates and by $Z = 0$ the hyperplane at infinity.

2.3 Bifurcation set

Let U be an affine manifold and $f : U \to \mathbb{C}$ be a non constant regular function. There exists a finite set $B \subset \mathbb{C}$ such that

$$f : U - f^{-1}(B) \to \mathbb{C} - B$$

is a locally trivial fibration. The smallest such set, denoted by $B(f)$, is called *the bifurcation set* of f and its points are called the *atypical values*. A value which is not atypical is typical. This set describes also the singular points of the Gauss-Manin system M of f. In general $B(f) = C(f) \cup B_\infty(f)$ where $C(f)$ is the set of critical values of f and $B_\infty(f)$ is a contribution of singular points at infinity. Keeping the previous notations, one has $B_\infty(f) \subset T_\infty(f)$ for a polynomial f and one should expect $B_\infty(f) \subset T_\infty^{fin}(f) \cup T_\infty^\infty(f)$ for a Laurent polynomial f.

One can be more precise if f has isolated singularities including at infinity. Keep the notations of section 2.2 and recall the number ν_a defined by (2). The next result refines equation (7):

Proposition 2.3.1 *[28, Theorem 4.12] Let f be a (Laurent) polynomial with isolated singularities including at infinity. Then a is typical if and only if $\nu_a = \mu_a = 0$.*

In particular, $B(f) = C(f)$ if f is cohomologically tame.

3 Applications to Gauss-Manin systems and their Fourier transform

We study here the Gauss-Manin systems of regular functions and their Brieskorn modules (of course, we have in mind Hori-Vafa models). As before, let U be an affine manifold of dimension $n \geq 2$, $S = \mathbb{C}$ and $f : U \to S$ be a regular function.

3.1 Gauss-Manin systems of regular functions

Let $\Omega^p(U)$ be the space of regular p-forms on U. The Gauss-Manin complex of f is

$$(\Omega^{\bullet+n}(U)[\partial_t], d_f)$$

where d_f is defined by

$$d_f(\sum_i \omega_i \partial_t^i) = \sum_i d\omega_i \partial_t^i - \sum_i df \wedge \omega_i \partial_t^{i+1}$$

The Gauss-Manin systems of f are the cohomology groups $M^{(i)}$ of this complex. These are holonomic regular $\mathbb{C}[t] < \partial_t >$-modules, see [Bo, p. 308], the action of t and ∂_t coming from the one on $\Omega^{\bullet+n}(U)[\partial_t]$ defined by

$$t(\sum_i \omega_i \partial_t^i) = \sum_i f\omega_i \partial_t^i - \sum_i i\omega_i \partial_t^{i-1}$$

and

$$\partial_t(\sum_i \omega_i \partial_t^i) = \sum_i \omega_i \partial_t^{i+1}$$

Lemma 3.1.1 *Assume that f has isolated singularities including at infinity. Then the modules $M^{(i)}$ are $\mathbb{C}[t]$ free of rank $h^{n-1+i}(U)$ for $i < 0$.*

Proof. Follows from equation (4) and the fact that $\mathcal{H}^{-1}({}^p\mathcal{H}^i(Rf_*\mathbb{C}_U[n]))$ is a constant sheaf on S for $i \neq n$ if f has isolated singularities including at infinity, see section 2.1. □

In general, we will put $M := M^{(0)}$ and we will call it *the Gauss-Manin system* of f. Let $\widehat{M}$ be its Fourier transform: this is M seen as a $\mathbb{C}[\tau] < \partial_\tau >$-module where τ acts as ∂_t and ∂_τ acts as $-t$. In particular

$$\widehat{M} = \frac{\Omega^n(U)[\tau]}{d_f(\Omega^{n-1}(U)[\tau])}$$

where $d_f(\sum_i \omega_i \tau^i) = \sum_i d\omega_i \tau^i - \sum_i df \wedge \omega_i \tau^{i+1}$. Let

$$G := \widehat{M}[\tau^{-1}] = \frac{\Omega^n(U)[\tau, \tau^{-1}]}{d_f(\Omega^{n-1}(U)[\tau, \tau^{-1}])}$$

be the localized module. Since M is a regular holonomic $\mathbb{C}[t] < \partial_t >$-module, G is a free $\mathbb{C}[\tau, \tau^{-1}]$-module equipped with a connection whose singularities are 0 and ∞ only, the former being regular and the latter of Poincaré rank less or equal to 1, see [24, V, prop. 2.2].

Recall that the rank of M is $\dim_{\mathbb{C}(t)} \mathbb{C}(t) \otimes_{\mathbb{C}[t]} M$, and this is also equal to the rank of $\mathbb{C}[t, p^{-1}(t)] \otimes_{\mathbb{C}[t]} M$, $p^{-1}(0)$ being the set of the singular points of M.

Theorem 3.1.2 *If f has at most isolated singularities including at infinity one has*

$$\operatorname{Rank} M = \mu + \nu + h^{n-1}(U) - h^n(U) \tag{12}$$

and

$$\operatorname{Rank} G = \mu + \nu \tag{13}$$

where μ is the global Milnor number of f, see equation (1).

Proof. By formula (5) one has $DR(\mathcal{M}) = (R^{n-1}f_*\mathbb{C}_U)[1]$ and it follows that $\mathcal{M}_a$ is a free $\mathcal{O}_a$-module of rank $\dim H^{n-1}(f^{-1}(a), \mathbb{C})$ for $a \notin p^{-1}(0)$. But $\mathcal{O}_a \otimes_{\mathbb{C}[t]} M$ is also isomorphic to $(\mathcal{O}_a)^{\operatorname{Rank} M}$ and it follows that the rank of M is equal to $\dim H^{n-1}(f^{-1}(a), \mathbb{C})$. The first formula then follows from proposition 2.1.2. For the second, we use the exact sequence

$$... \to M^{(j)} \xrightarrow{\partial_t} M^{(j)} \to H^{n+j}(U, \mathbb{C}) \to \cdots \to H^{n-1}(U, \mathbb{C}) \to M \xrightarrow{\partial_t} M \to H^n(U, \mathbb{C}) \to 0$$

for $j \leq 0$. If f has isolated singularities including at infinity, it follows from lemma 3.1.1 that ∂_t is surjective on $M^{(-1)}$ and this gives the exact sequence

$$0 \to H^{n-1}(U, \mathbb{C}) \to M \xrightarrow{\partial_t} M \to H^n(U, \mathbb{C}) \to 0$$

We also have

$$\operatorname{Rank} G = \operatorname{Rank} M + \dim(\operatorname{coker} \partial_t) - \dim(\ker \partial_t)$$

see for instance [24, Proposition V.2.2], and the second formula follows from the first one. □

3.2 Slopes

We use here the terminology of [18]. Notice the following properties of G:

- G has no ramification because M is regular, see for instance [24, V. 3. b.]. In particular G has only integral slopes, the slopes 0 and 1.
- If H is a lattice in G, *i.e* a free $\mathbb{C}[\theta]$-module of maximal rank, stable under $\theta^2\partial_\theta$, the eigenvalues of the constant matrix in the expression of $\theta^2\partial_\theta$ in a basis of H are precisely the singular points of the Gauss-Manin system M, see [24, V. 3].

The condition "no ramification" is a characteristic property of the Fourier transform of regular holonomic modules, see *f.i* [25, lemma 1.5]. In order to emphasize it, let us consider the following example: let M be a meromorphic connection of rank 3 and $(\omega_0, \omega_1, \omega_2)$ a basis of M over $\mathbb{C}[\theta, \theta^{-1}]$ in which the system takes the form

$$\theta^2\partial_\theta = A_0 + A_1\theta$$

where

$$A_0 = \begin{pmatrix} L_2^1 q & L_1^2 q^2 & L_0^3 q^3 \\ 1 & L_1^1 q & L_0^2 q^2 \\ 0 & 1 & L_0^1 q \end{pmatrix}$$

and $A_1 = \mathrm{diag}(0,1,2)$. The section ω_0 is cyclic and its minimal polynomial Q can have rational slopes (possible cases 1/2, 1/3 et 2/3) and integral slopes (possible cases 0 and 1), depending on the values of the coefficients of the matrix A_0. Assume moreover that

$$L_0^1 = 60,\ L_1^1 = 312,\ L_2^1 = 60,\ L_0^2 = 20520,\ L_1^2 = 20520 \text{ et } L_0^3 = 1339200$$

This is the differential system associated with the small quantum cohomology of a smooth hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$, see section 7.2.2. Then Q has only two slopes, 0 and 1.

3.3 Brieskorn module

The Brieskorn module G_0 of the regular function f on U is by definition the image in G of the sections that do not depend on τ. Putting $\theta := \tau^{-1}$, we have

$$G_0 := \frac{\Omega^n(U)[\theta]}{d_f(\Omega^{n-1}(U)[\theta,\theta^{-1}]) \cap \Omega^n(U)[\theta]}$$

where

$$d_f(\sum_i \omega_i\theta^i) = \sum_i [d\omega_i\theta^{i+1} - df \wedge \omega_i\theta^i].$$

If f is cohomologically tame, this module provides a lattice in G, that is a free $\mathbb{C}[\theta]$-module of maximum rank [23]. This result is no longer true in general. Recall the global Milnor number μ defined in equation (1).

Proposition 3.3.1 *Assume that f has only isolated critical points on U.*

1. One has

$$G_0 = \frac{\Omega^n(U)[\theta]}{(\theta d - df\wedge)\Omega^{n-1}(U)[\theta]}$$

and

$$\frac{G_0}{\theta G_0} = \frac{\Omega^n(U)}{df \wedge \Omega^{n-1}(U)}$$

this vector space being of dimension μ.

2. The $\mathbb{C}[\theta]$-module G_0 has no torsion.

3. Sections of G_0 are linearly independent over $\mathbb{C}[\theta]$ if their classes are independent in $G_0/\theta G_0$.

Proof. For assertions 1. and 2. we use the classical generalized de Rham lemma: if f has only isolated critical points on U, the cohomology groups of the complex $(\Omega^\bullet(U),\ df\wedge)$ all vanish, except possibly the one in degree n which is equal to

$$\frac{\Omega^n(U)}{df \wedge \Omega^{n-1}(U)}$$

In order to show 3., let us assume that $\sum_{i=1}^{\mu} a_i(\theta)\omega_i = 0$ in G_0. If the classes of the ω_i's are independent in $G_0/\theta G_0$, we first get $a_i(0) = 0$ for all i. Using the fact that G_0 has no $\mathbb{C}[\theta]$-torsion, we also get that the coefficients of the monomials θ^k in the $a_i(\theta)$'s all vanish. □

Corollary 3.3.2 *Assume that f has only isolated critical points U. Then*

1. $\operatorname{Rank} G \geq \mu$.

2. $\operatorname{Rank} G = \mu$ *if G_0 is free of finite type.*

Proof. 1. By proposition 3.3.1 there exists a free module of rank μ in G. 2. If G_0 is free of finite type, it follows from proposition 3.3.1 that its rank is μ. In this case, G_0 is a lattice in G and thus $\operatorname{Rank} G = \mu$. □

As a consequence, G_0 will not be of finite type if $\operatorname{Rank} G > \mu$. This happens for instance if f has isolated singularities including at infinity, see theorem 3.1.2. Notice also that the converse of point 2 is true if we assume moreover that f has isolated singularities including at infinity, see [23].

3.4 Basic example

We test the previous results on a classical wild example [1]. Let f be defined on $\mathbb{C}^2$ by

$$f(x, y) = y(xy - 1)$$

It has no critical points at finite distance.

Proposition 3.4.1 *1. f has only one singular point at infinity. The number ν of vanishing cycles at infinity is equal to* 1 *and $B(f) = \{0\}$.*

2. The $\mathbb{C}[\tau,\tau^{-1}]$-module G is free of rank 1 and the class $[dx\wedge dy]$ of $dx\wedge dy$ is a basis of it.

Proof. 1. This result is well-known but we give the proof in order to set the notations. Let us keep the notations of section 2.2. Homogeneization of the fibers of f gives

$$F(X,Y,Z,t)=XY^2-YZ^2-tZ^3=0$$

where the equation $Z=0$ defines the hyperplane at infinity. Notice that $\mathbb{Y}_{sing}=\{p\}\times\mathbb{C}$ where $p=(1:0:0)$ and in order to compute the number of vanishing cycles at infinity we can use corollary 2.2.2. The Milnor number of the singularity $u^2-uv^2-tv^3=0$ at $(0,0)$ is equal to 2 for all $t\neq 0$ and is equal to 3 for $t=0$. The point p is thus an isolated singular point of f at infinity and we have $\nu_{p,0}=1$ [2].
2. By 1. and theorem 3.1.2, we know that G is free of rank 1 over $\mathbb{C}[\tau,\tau^{-1}]$. The differential form

$$\omega=rx^{r-1}y^p dx+px^r y^{p-1}dy,$$

$r,p\geq 1$, is exact. We thus have $[df\wedge\omega]=0$ and

$$[(2r-p)x^r y^{p+1}dx\wedge dy]=[rx^{r-1}y^p dx\wedge dy] \tag{14}$$

in G for $r,p\geq 1$. An analogous computation shows that $[y^{p+1}dx\wedge dy]=0$ if $p\geq 1$ and that $[2rx^r ydx\wedge dy]=[rx^{r-1}dx\wedge dy]$ if $r\geq 1$. If $2r\neq p$, one can express in particular $[x^r y^{p+1}dx\wedge dy]$ in terms of $[x^{r-1}y^p dx\wedge dy]$. If $2r=p$, notice that

$$\tau[x^r y^{2r+1}dx\wedge dy]=[x^r y^{2r}dx\wedge dy] \tag{15}$$

Indeed, $df\wedge x^r y^{2r+1}dx=(-2x^{r+1}y^{2r+2}+x^r y^{2r+1})dx\wedge dy$ hence

$$(2r+1)[x^r y^{2r}dx\wedge dy]=2\tau[x^{r+1}y^{2r+2}dx\wedge dy]-\tau[x^r y^{2r+1}dx\wedge dy]$$

and we get formula (15) using formula (14). This comptutation holds also for $r=0$, in particular $\tau[ydx\wedge dy]=[dx\wedge dy]$. Last,

$$\tau^{-1}[x^q dx\wedge dy]=[df\wedge\frac{x^{q+1}}{q+1}dy]=[y^2\frac{x^{q+1}}{q+1}dx\wedge dy]=[a_q x^q ydx\wedge dy]=[b_q x^{q-1}dx\wedge dy]$$

for $q\geq 1$, where a_q and b_q are non zero constant, as shown by formula (14). These observations show that one can express the class of any form in terms of $[dx\wedge dy]$, which is thus a generator of G. □

We will consider other wild examples in section 5.3.

4 Hypersurfaces in weighted projective spaces

In this section we recall basic results about hypersurfaces in weighted projective spaces. We will consider only *smooth* hypersurfaces and the goal of this section is to give a characterization of such objects, see theorem 4.1.3. Our references are [4], [7] and [17].

[2] To make the link with remark 2.2.4, notice that $f(n,\frac{1}{2n})\to 0$ and $\operatorname{grad} f(n,\frac{1}{2n})\to(0,0)$ so that $0\in T_\infty(f)$.

4.1 Smooth hypersurfaces in weighted projective spaces

Let $w_0, \cdots, w_n$ and d be integers greater than zero. In what follows, except otherwise stated, we will assume that $n \geq 3$ and that the weighs w_i are normalized, that is

$$P.G.C.D.(w_0, \cdots, \hat{w}_i, \cdots, w_n) = 1 \text{ for all } i = 0, \cdots, n \text{ and } w_0 \leq w_1 \leq \cdots \leq w_n \tag{16}$$

Recall that a polynomial W is quasi-homogeneous of weight $(w_0, \cdots, w_n)$ and of degree d if

$$W(\lambda^{w_0} u_0, \cdots, \lambda^{w_n} u_n) = \lambda^d W(u_0, \cdots, u_n)$$

for any non zero λ. Equation $W(u_0, \cdots, u_n) = 0$ defines a hypersurface H (resp. CH) of degree d in the weighted projective space $\mathbb{P}(w) := \mathbb{P}(w_0, \cdots, w_n)$ (resp. $\mathbb{C}^{n+1}$). The hypersurface H is *quasi-smooth* if $CH - \{0\}$ is smooth.

Example 4.1.1 *If $d = w_i$ for some index i then $W = a_i u_i + g(u_0, \cdots, \widehat{u_i}, \cdots, u_n)$, where $a_i \in \mathbb{C}^*$ and g is quasi-homogeneous. H is then quasi-smooth and isomorphic to the weighted projective space $\mathbb{P}(w_0, \cdots, \widehat{w_i}, \cdots, w_n)$ via the isoomorphism*

$$(u_0, \cdots, \widehat{u_i}, \cdots, u_n) \mapsto (u_0, \cdots, -a_i^{-1} g(u_0, \cdots, \widehat{u_i}, \cdots, u_n), \cdots, u_n)$$

In this case, we will say that H is a linear cone.

Let $\mathbb{P}_{sing}(w)$ be the singular locus of $\mathbb{P}(w)$. The hypersurface H is in *general position* with respect to $\mathbb{P}_{sing}(w)$ (for short: in general position) if

$$\operatorname{codim}_H(H \cap \mathbb{P}_{sing}(w)) \geq 2 \tag{17}$$

A hypersurface in general position inherits the singularities of the ambient space:

Proposition 4.1.2 *1. Assume that the degree d hypersurface H is in general position and quasi-smooth. Then*

$$\omega_H \simeq \mathcal{O}_H(d - \sum_{i=0}^{n} w_i) := \mathcal{O}_{\mathbb{P}(w)}(d - \sum_{i=0}^{n} w_i)_{|H}$$

where ω_H denotes the canonical bundle. One has also $Pic(H) = \mathbb{Z}$.

2. The singular locus of a quasi-smooth hypersurface H in general position is $H_{sing} = H \cap \mathbb{P}_{sing}(w)$.

Proof. See [7, Theorem 3.3.4 and Theorem 3.2.4] for 1. and [4, Proposition 8] for 2. □

Put

$$w := \sum_{i=0}^{n} w_i \tag{18}$$

Under the assumptions of proposition 4.1.2, we will say that H is *Fano* if $d < w$ and *Calabi-Yau* if $d = w$. We will mainly consider the Fano case.

We will use the following characterization of smooth hypersurfaces in section 5:

Theorem 4.1.3 *Let H be a degree d hypersurface in $\mathbb{P}(w_0, \cdots, w_n)$. Assume that* [3]

1. *P.G.C.D.$(w_i, w_j) = 1$ for all i, j,*
2. *w_i divides d for all i,*
3. *$w_i < d$ for all i.*

Then H is not a linear cone, is in general position, quasi-smooth and smooth.

Proof. By [17, I.3.10], a degree d hypersurface is in general position if and only if

$$P.G.C.D.(w_0, \cdots, \hat{w}_i, \cdots, \hat{w}_j, \cdots, w_n)|d$$

for all i, j, $i \neq j$, and

$$P.G.C.D.(w_0, \cdots, \hat{w}_i, \cdots, \cdots, w_n) = 1$$

for all i. Therefore the first condition shows that H is in general position. The second condition shows that H is quasi-smooth, see [17, Theorem I.5.1]. Last, and in order to show that H is smooth we use the following numerical criterion [4]: for any prime p, let us define

$$m(p) = card\{i; p \text{ divides } w_i\}, \; k(p) = 1 \text{ if } p \text{ divides } d, \; 0 \text{ otherwise}, \; q(p) = n - m(p) + k(p) \quad (19)$$

Then the quasi-smooth and in general position degree d hypersurface H is smooth if and only if $q(p) \geq n$ for any prime p. The first condition shows that $m(p) \leq 1$: if $m(p) = 0$ we get, by the very definition, $q(p) \geq n$; if $m(p) = 1$ the second condition shows that $k(p) = 1$ and thus $q(p) = n$. □

Example 4.1.4 *(Surfaces) The degree 6 hypersurface in $\mathbb{P}(1, 1, 2, 3)$ is in general position and smooth. It's a Fano surface. The other smooth Fano surfaces are the surfaces of degree 2 or 3 in $\mathbb{P}(1, 1, 1, 1)$ and surfaces of degree 4 in $\mathbb{P}(1, 1, 1, 2)$.*

Remark 4.1.5 *(Curves) The previous results have been established for $n \geq 3$. If H is a curve of degree d in $\mathbb{P}(w_0, w_1, w_2)$ then H is in general position, is smooth and is not a linear cone if and only if the conditions of theorem 4.1.3 are satisfied [17, Theorem II.2.3].*

4.2 The quantum differential equation of a smooth hypersurface

Let H be a degree d smooth hypersurface in the weighted projective space $\mathbb{P}(w_0, \cdots, w_n)$. The differential operator

$$P_H(\theta q\partial_q, q, \theta) = \prod_{i=0}^{n}(w_i\theta q\partial_q)(w_i\theta q\partial_q - \theta)\cdots(w_i\theta q\partial_q - (w_i - 1)\theta) - q(d\theta q\partial_q + \theta)\cdots(d\theta q\partial_q + d\theta) \quad (20)$$

(q is a quantum variable) is called the *quantum differential operator* of H. We will often write P instead of P_H. We will call

$$P_H(\theta q\partial_q, q, \theta) = 0 \quad (21)$$

[3]The first and the second conditions imply the third except when H is a degree d hypersurface in $\mathbb{P}(1, \cdots, 1, d)$: the purpose of the third condition is to remove the linear cones. This will simplify the statements.

the quantum differential equation. The key point is that the quantum differential equation, which depends only on combinatorial data, can be used in order to describe the small quantum cohomology of the H, see for instance [2] and section 7.2.1.

Let us define

$$M_A = \mathbb{C}[\theta, q, q^{-1}] < \theta q \partial_q > / \mathbb{C}[\theta, q, q^{-1}] < \theta q \partial_q > P_H \tag{22}$$

This is a $\mathbb{C}[\theta, q, q^{-1}]$-module of finite type.

Proposition 4.2.1 *Under the assumptions of theorem 4.1.3, M_A is a $\mathbb{C}[\theta, q, q^{-1}]$-module of rank n.*

Proof. Notice first that, using the relation $\partial_q q = q\partial_q + 1$, equation (21) takes the form

$$\theta^{\mu} \prod_{i=0}^{n} w_i^{w_i} \prod_{i=0}^{n} (q\partial_q)(q\partial_q - \frac{1}{w_i}) \cdots (q\partial_q - \frac{w_i - 1}{w_i}) = \theta^d d^d (q\partial_q)(q\partial_q - \frac{1}{d}) \cdots (q\partial_q - \frac{d-1}{d}) q \tag{23}$$

By assumption, w_i divides d: we write $d = m_i w_i$ and we define

$$v_i := card\{k \in \{1, \cdots, d-1\};\ m_i \text{ divides } k\}$$

for $i = 0, \cdots, n$. Let $k \in \{1, \cdots, d-1\}$. If m_i divides k, write $k = m_i \ell_i$: we have $d\ell_i = k w_i$ and thus $\frac{\ell_i}{w_i} = \frac{k}{d}$. Conversely, if there exists $k \in \{1, \cdots, d-1\}$ such that $\frac{\ell_i}{w_i} = \frac{k}{d}$ then $k = m_i \ell_i$. Using (23) we see that, after cancellation of the common factors on the left and on the right, the quantum differential operator P_H is of degree $w_0 + \cdots + w_n - 1 - \sum_{i=0}^{n} v_i$ in $q\partial_q$. If $d = w_1 \cdots w_n$ we have $v_i = w_i - 1$ for $i = 1, \cdots, n$ and the proposition follows because the rank of M_A is the degree of the irreducible polynomial P in $\theta q \partial_q$. $\square$

5 Hori-Vafa models

We define here, following [13] and [16], mirror partners for the small quantum cohomology of smooth hypersufaces in weighted projective spaces. Let H be a degree d hypersurface in $\mathbb{P}(w_0, \cdots, w_n)$. Except otherwise stated, we assume that

$$d \leq w - 1 := w_0 + w_1 + \cdots + w_n - 1$$

which is precisely the Fano condition of section 4.1.

5.1 Hori-Vafa models as Laurent polynomials

The Hori-Vafa model of H (for short: H-V model) is the function f defined on the variety U where:

1. $f = u_0 + \cdots + u_n$,
2. U is defined by the equations

$$\begin{cases} u_0^{w_0} \cdots u_n^{w_n} = q \\ \sum_{j \in J} u_j = 1 \end{cases} \tag{24}$$

where J is a set of indices such that $\sum_{j \in J} w_j = d$.

Here q is the quantization variable. The following result is [22, Theorem 9]:

Proposition 5.1.1 ([22]) *Under the assumptions of theorem 4.1.3, one may assume that*

$$w_0 = 1 \ \text{et} \ d = w_{r+1} + \cdots + w_n$$

for some $r \in \{0, \cdots, n-2\}$. In these conditions, the Hori-Vafa model of H takes the form, for $(x_1, \cdots, x_{n-1}) \in (\mathbb{C}^*)^{n-1}$,

$$f(x_1, \cdots, x_{n-1}) = x_1 + \cdots + x_r + 1 + q\frac{(x_{r+1} + \cdots + x_{n-1} + 1)^d}{x_1^{w_1} \cdots x_{n-1}^{w_{n-1}}} \tag{25}$$

if $r \geq 1$ (that is $d \leq w - 2$) and

$$f(x_1, \cdots, x_{n-1}) = 1 + q\frac{(x_1 + \cdots + x_{n-1} + 1)^d}{x_1^{w_1} \cdots x_{n-1}^{w_{n-1}}}$$

if $r = 0$ (that is $d = w - 1$). □

Remark 5.1.2 *By [22, Proposition 7] and under the assumptions of theorem 4.1.3, there are at least $w - d + 1$ weights w_i equal to 1. It follows that*

$$n + d > w \tag{26}$$

We will see in sections 5.3 and 6 that $n+d-w$ is a potential number of vanishing cycles at infinity.

For the two next results, we fix $q = q_0 \in \mathbb{C}^*$. We denote by f^o the Laurent polynomial (25) for $q = q_0$ and by Q^o_f its Jacobian ring.

Lemma 5.1.3 *The Laurent polynomial f^o has $w - d$ isolated, non degenerate, critical points on $(\mathbb{C}^*)^{n-1}$. These points are defined by*

$$c_k = (b_1\varepsilon^k, \cdots, b_r\varepsilon^k, \frac{w_{r+1}}{w_n}, \cdots, \frac{w_{n-1}}{w_n}) \tag{27}$$

for $k = 0, \cdots, w - d - 1$ where ε denotes a $w - d$-th primitive root of the unity and $b_i = w_i(q_0\frac{d^d}{w_1^{w_1} \cdots w_n^{w_n}})^{1/(w-d)}$ for $i = 1, \cdots, r$. The corresponding critical values are

$$f^o(c_k) = (w - d)(q_0\frac{d^d}{w_1^{w_1} \cdots w_n^{w_n}})^{1/(w-d)}\varepsilon^k + 1 \tag{28}$$

for $k = 0, \cdots, w - d - 1$ and we have $\prod_{k=0}^{w-d-1} f^o(c_k) = (w - d)^{w-d}q_0\frac{d^d}{w_1^{w_1} \cdots w_n^{w_n}}$

Proof. Direct computations. □

Corollary 5.1.4 *1. The eigenvalues of the multiplication by f^o on Q^o_f are pairwise distinct,*

2. the classes of $1, f^o, \cdots, (f^o)^{w-d-1}$ provide a basis of Q^o_f,

3. one has $(f^o)^{w-d} = (w - d)^{w-d}q_0\frac{d^d}{w_1^{w_1} \cdots w_n^{w_n}}$.

Proof. The critical values of f are pairwise distinct by lemma 5.1.3 and their product is equal to the right hand side of the last equality. □

5.2 A homogeneous version of H-V models and its relative Gauss-Manin system

For $i = r+1, \cdots, n-1$, let us define $u_i = q^{1/w_n} x_i$ and $u_i = x_i$ if $i = 1, \cdots, r$. Then formula (25) takes the form, putting $Q := q^{1/w_n}$ and removing the additive constant 1,

$$f(u_1, \cdots, u_{n-1}, Q) = u_1 + \cdots + u_r + \frac{(u_{r+1} + \cdots + u_{n-1} + Q)^d}{u_1^{w_1} \cdots u_{n-1}^{w_{n-1}}} \tag{29}$$

if $r \geq 1$, this formula being easily adapted for $r = 0$. Results of lemma 5.1.3 remain unchanged (replace q by Q^{w_n}). From now on, we will use this description the reason being the following homogeneity relation

$$f = \frac{w-d}{w_n} Q \frac{\partial f}{\partial Q} + \sum_{i=1}^{r} u_i \frac{\partial f}{\partial u_i} + \frac{w-d}{w_n} \sum_{i=r+1}^{n-1} u_i \frac{\partial f}{\partial u_i} \tag{30}$$

from which it follows in particular that

$$u_1, \cdots, u_r \text{ are of degree } 1 \tag{31}$$

and

$$u_{r+1}, \cdots, u_{n-1} \text{ et } Q \text{ are of degree } \frac{w-d}{w_n}. \tag{32}$$

The (localized Fourier transform) Gauss-Manin system G of (29) is defined as section 3: it is a free $\mathbb{C}[\theta, \theta^{-1}, q, q^{-1}]$-module equipped with a connection ∇ defined by

$$\theta^2 \nabla_{\partial_\theta} [\sum_i \omega_i \theta^i] = [\sum_i f \omega_i \theta^i] - [\sum_i i \omega_i \theta^{i+1}] \tag{33}$$

and

$$\theta \nabla_{Q\partial_Q} [\sum_i \omega_i \theta^i] = [\sum_i Q \partial_Q(\omega_i) \theta^{i+1}] - [\sum_i Q \frac{\partial f}{\partial Q} \omega_i \theta^i] \tag{34}$$

where the ω_i's are differential forms on $(\mathbb{C}^*)^{n-1} \times \mathbb{C}^*$, equipped with coordinates $(u_1, \cdots, u_{n-1}, Q)$, $Q\partial_Q(\omega_i)$ denotes the Lie derivative of the differential form ω_i in the direction of $Q\partial_Q$ and [ ] denotes the class in G.

The following result relies the actions of $\theta^2 \nabla_{\partial_\theta}$ and $\theta \nabla_{Q\partial_Q}$ in G. Let $(a_1, \cdots, a_{n-1}) \in \mathbb{Z}^{n-1}$,

$$\omega_0 = \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{n-1}}{u_{n-1}} \tag{35}$$

and $[u_1^{a_1} \cdots u_{n-1}^{a_{n-1}} \omega_0]$ be the class of $u_1^{a_1} \cdots u_{n-1}^{a_{n-1}} \omega_0$ in G.

Lemma 5.2.1 *One has*

$$\theta^2 \nabla_{\partial_\theta} [u_1^{a_1} \cdots u_{n-1}^{a_{n-1}} \omega_0] = -\frac{w-d}{w_n} \theta \nabla_{Q\partial_Q} [u_1^{a_1} \cdots u_{n-1}^{a_{n-1}} \omega_0] + (\sum_{i=1}^{r} a_i + \frac{w-d}{w_n} \sum_{i=r+1}^{n-1} a_i) \theta [u_1^{a_1} \cdots u_{n-1}^{a_{n-1}} \omega_0] \tag{36}$$

in G.

Proof. Follows from (30) and the definition of ∇. □

5.3 The rank of G

How to compute the rank of the connection G associated with the H-V model f defined in section 5.2? The hope is to use theorem 3.1.2, especially formula (13): the main question is to decide whether f has at most isolated singularities including at infinity for a suitable compactification or not. This problem is in general very difficult[4]. Let us begin with the following examples:

Example 5.3.1 *1. Let us consider the H-V model[5] of a smooth hypersuface of degree 2 in $\mathbb{P}^3$:*

$$f(x,y) = x + \frac{(y+1)^2}{xy}$$

Keep the notations of section 2.2 and remark 2.2.3: the equation $F(X,Y,Z,t) = 0$ takes the form

$$X^2Y - tXYZ + Z(Y+Z)^2 = 0$$

and $\mathbb{Y}_{sing} = P \times \mathbb{C}$ where $P = (0 : -1 : 1)$ is on the polar locus at finite distance (the hyperplane at infinity has the equation $Z = 0$). In order to compute the number of vanishing cycles $\nu_{P,t}$, we use corollary 2.2.2: the hypersurface

$$u^2v - tuv + (1+v)^2 = 0$$

is smooth for $t \neq 0$ but the Milnor number at P for $t = 0$ is $\mu_{P,0} = 1$. Thus $\nu_{P,0} = \mu_{P,0} = 1$. The value $t = 0$ is atypical. By theorem 3.1.2, the rank of G is $w - d + \nu = 2 + 1 = 3$. Notice that the set $T_\infty(f)$ defined in remark 2.2.4 is void despite the fact that f is not cohomologically tame (see the discussion of remark 2.2.4).

2. Let us now consider the H-V model of a smooth hypersurface of degree 2 in $\mathbb{P}^4$:

$$f(x,y,z) = x + y + \frac{(z+1)^2}{xyz}$$

With the notations of section 2.2, the equation $F(X_0, X_1, X_2, X_3, t) = 0$ takes the form

$$X_1^2X_2X_3 + X_1X_2^2X_3 + X_0^2(X_3 + X_0)^2 - tX_0X_1X_2X_3 = 0$$

and we check as above that:

- *f has no singular point on the hyperplane at infinity $X_0 = 0$,*
- *f has an isolated singular point $P = (1 : 0 : 0 : -1)$ on the polar locus at finite distance for which $\nu_{P,0} = 4 - 3 = 1$ as it follows from proposition 2.2.1.*

P is thus a singular point at infinity, the value $t = 0$ is atypical and the number of vanishing cycles at infinity is 1. By theorem 3.1.2, the rank of G is therefore 4.

[4] More generally, there exist different theoretic classes of functions having isolated singularities including at infinity in some sense, see for instance the book [30] and the references therein. But in general one cannot decide if a given function belongs to a class or to another.

[5] We fix here $q = 1$.

Example 5.3.2 *The H-V model of a degree d hypersurface in $\mathbb{P}^n$ takes the form*

$$f(u_1, \cdots, u_{n-1}) = u_1 + \cdots + u_{n-d} + \frac{(u_{n-d+1} + \cdots + u_{n-1} + q)^d}{u_1 \cdots u_{n-1}}$$

Recall the sets $T_\infty^{fin}(f)$, $T_\infty^\infty(f)$ and $T_\infty(f)$ defined in remark 2.2.4. We have

$$T_\infty^{fin}(f) = \{0\} \text{ et } T_\infty^\infty(f) = T_\infty(f) = \emptyset.$$

Indeed, let us define the sequence $(u_p) = ((u_1^p, \cdots, u_{n-1}^p))$ by

$$u_1^p = \cdots = u_{n-d}^p = \frac{1}{p} \text{ et } u_{n-d+1}^p = \cdots = u_{n-1}^p = \frac{1}{p^{n-d}} - \frac{q}{d-1}$$

Then

$$u_p \to (0, \cdots, 0, -\frac{q}{d-1}, \cdots, -\frac{q}{d-1}),\ u_p \operatorname{grad} f(u_p) \to 0 \text{ et } f(u_p) \to 0$$

thus $\{0\} \subset T_\infty^{fin}(f)$, see section 2.3. The same kind of computations shows that there are no other candidates in $T_\infty^{fin}(f)$ and that $T_\infty^\infty(f) = T_\infty(f) = \emptyset$.

This suggests the following conjectures, which will be emphasized by the discussion in section 6.1 below:

Conjecture 5.3.3 *(Optimistic) Under the assumptions of theorem 4.1.3, there exists a compactification for which Hori-Vafa models have only one singular point P at infinity, located on the polar locus at finite distance and such that $\nu = \nu_{P,0} = n + d - w$.*

Notice that $n + d - w > 0$ under the assumptions of theorem 4.1.3, see remark 5.1.2. Conjecture 5.3.3 has been verified in example 5.3.1 using the standard compactification.

Conjecture 5.3.4 *(Realistic) Under the assumptions of theorem 4.1.3 the rank of G is equal to n.*

It follows from theorem 3.1.2 and lemma 5.1.3 that conjecture 5.3.3 implies conjecture 5.3.4.

Corollary 5.3.5 *Under the assumptions of theorem 4.1.3, if conjecture 5.3.4 holds true then the Brieskorn module of a Hori-Vafa model is not of finite type (in particular, a H-V model is not cohomologically tame).*

Proof. Follows from lemma 5.1.3 and corollary 3.3.2 because $w - d < n$ under the assumptions of theorem 4.1.3, see remark 5.1.2 . □

6 Application to mirror symmetry for smooth hypersurfaces in projective spaces. The case of the quadrics

We explain in this section why the Hori-Vafa models should be mirror partners of smooth hypersurfaces in weighted projective spaces. The general setting is described in section 6.1 and we apply it to quadrics in $\mathbb{P}^n$ in section 6.2. We work in the sequel under the assumptions of theorem 4.1.3.

6.1 Mirror symmetry and the Birkhoff problem

Let H be a smooth Fano degree d hypersurface in the weighted projective space $\mathbb{P}(1, w_1, \cdots, w_n)$. The general principle is to show that the quantum differential operator P_H defined in section 4.2 is a minimal polynomial of a section in G (see section 5.2) of a suitable Hori-Vafa model, see equation (29). This can be done solving the following Birkhoff problem for the H-V model alluded to: find a free $\mathbb{C}[q, \theta]$-module H_0^{log} of rank n in G and a basis $(\omega_0, \cdots, \omega_{n-1})$ of it in which the matrix of the flat connection ∇ takes the form

$$(\frac{A_0(q)}{\theta} + A_1(q))\frac{d\theta}{\theta} + (\frac{\Omega_0(q)}{\theta} + \Omega_1(q))\frac{dq}{q} \tag{37}$$

and such that

$$P(\theta\nabla_{q\partial_q}, q, \theta)(\omega_0) = 0 \tag{38}$$

where $P(\theta\nabla_{q\partial_q}, q, \theta)$ is defined by equation (20). We also require that $A_1(q)$ is semi-simple, with eigenvalues $0, 1, \cdots, n-1$ (and, up to a factor 2, this corresponds to cohomology degrees). Notice that, unlike the absolute case, the expected module H_0^{log} is not the Brieskorn module G_0. The size of the matrices alluded to should be equal to n because of the degree of P, see proposition 4.2.1. It follows that the rank of G is greater or equal than n: conjecture 5.3.4 asserts that this rank is precisely equal to n.

As explained in remark 7.2.1, it follows from (38) that the matrix $A_0(q)$ (which is the matrix of multiplication by f on $H_0^{log}/\theta H_0^{log}$) provides the characteristic relation [6]

$$b^{\circ n} = q\frac{d^d}{\prod_{i=1}^n w_i^{w_i}} b^{\circ d+n-w} \tag{39}$$

in small quantum cohomology, where $\circ$ denotes the quantum product, b the hyperplane class and $w = w_0 + \cdots + w_n$, [2], [15], see section 7.2.2 for details. Therefore it deserves a particular study. Recall that $n + d - w > 0$, see remark 5.1.2. Let $P_c(A_0)$ be the characteristic polynomial of A_0.

Proposition 6.1.1 *Assume that the rank of G is equal to n. One has*

$$P_c(A_0)(\zeta, q) = P_c^{fin}(A_0)(\zeta, q) P_c^{\infty}(A_0)(\zeta, q) \tag{40}$$

where

$$P_c^{fin}(A_0)(\zeta, q) = \zeta^{w-d} - (w-d)^{w-d}\frac{d^d}{\prod_{i=0}^n w_i^{w_i}} q \tag{41}$$

and

$$P_c^{\infty}(A_0)(\zeta, q) = \zeta^{n+d-w} + \sum_{i,j} a_{i,j\geq 0}\zeta^i q^j \text{ with } i + (w-d)j = n + d - w \tag{42}$$

In particular,

$$P_c(A_0)(\zeta, q) = \zeta^n - (w-d)^{w-d}\frac{d^d}{\prod_{i=0}^n w_i^{w_i}} q\zeta^{n+d-w} \tag{43}$$

if and only if $B_\infty(f) = \{0\}$, see section 2.3.

[6] This formula is well known for hypersurfaces in $\mathbb{P}^n$, in which case $w = n+1$, see [3, 11.2.1], [13] and section 7.2.1.

Proof. For fixed q, different from 0, the eigenvalues of $A_0(q)$ are precisely the atypical values of f, with appropriate multiplicities, see section 3.2. Moreover, the coefficients of $A_0(q)$ are homogeneous in q: a coefficient $a_{r,s}$ is homogeneous of degree $s-r+1$, see section 5.2 (recall that q is homogeneous of degree $w-d$). It follows that the characteristic polynomial $P_c(A_0)(\zeta, q)$ of $A_0(q)$ is homogeneous of degree n, ζ being of degree 1. Therefore, equation (40) follows from lemma 5.1.3. For the last assertion, use the fact that the eigenvalues of $A_0(q)$ are the singular points of the classical Gauss-Manin system M, see section 3.2. □

Remark 6.1.2 *Formula (37) yield a polynomial Q in the variables $(\nabla_{\theta^2\partial_\theta}, q, \theta)$, which annihilates ω_0 and which gives informations about the irregularity of system (37), see for instance [18] and section 3.2 . Assume that the characteristic polynomial of $A_0(q)$ takes the form (43). Then Q has only two slopes,* 0 *and* 1 *and one has*

$$\mathrm{Rank}\, H^{log} = \mathrm{Irr}\, Q + \mathrm{Reg}\, Q$$

where the irregularity $\mathrm{Irr}\, Q$ *of Q is $w-d$ and its regularity* $\mathrm{Reg}\, Q$ *is $n+d-w$. Indeed, the Newton polygon of Q is the one of $(\theta^2\partial_\theta)^n - (w-d)^{w-d}\frac{d^d}{\prod_{i=0}^n w_i^{w_i}} q(\theta^2\partial_\theta)^{n+d-w}$. Notice that* $\mathrm{Irr}\, Q$ *is the dimension of the Jacobian ring and that* $\mathrm{Reg}\, Q$ *is the expected number of vanishing cycles at infinity, see section 5.3. If moreover ω_0 is cyclic,* $\mathrm{Irr}\, Q$ *and* $\mathrm{Reg}\, Q$ *are the regularity and the irregularity of H.*

6.2 Illustration: smooth quadrics in $\mathbb{P}^n$

The aim of this section is to test the previous discussions for quadrics in $\mathbb{P}^n$. This paragraph has been inspired by [14], which deals in a slightly different way with quadrics in $\mathbb{P}^4$. We prove[7] in particular the theorem announced in the introduction, see section 6.2.3. We do not use any conjecture in this section.

6.2.1 The Hori-Vafa model of a quadric

The Hori-Vafa model of a quadric in $\mathbb{P}^n$ is

$$f(u_1, \cdots, u_{n-1}) = u_1 + \cdots + u_{n-2} + \frac{(u_{n-1}+q)^2}{u_1 \cdots u_{n-1}}$$

The (localized Fourier transform of the) Gauss-Manin system G of f is a free $\mathbb{C}[\theta, \theta^{-1}, q, q^{-1}]$-module and is equipped with a connection ∇ whose covariant derivatives are defined by formulas (33) and (34), see section 5.2.

6.2.2 The Birkhoff problem

Let $\omega_0 = \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{n-1}}{u_{n-1}}$ and

$$\varepsilon := ([\omega_0], [u_1\omega_0], \cdots, [u_1 \cdots u_{n-2}\omega_0], 2[u_{n-1}\omega_0]) := (\varepsilon_0, \cdots, \varepsilon_{n-1}) \tag{44}$$

[7] We don't use any conjecture in this section.

where $[\alpha]$ denotes the class of α in G. Recall the Brieskorn module G_0 defined as in section 3.3. One has

$$\frac{G_0}{\theta G_0} = \frac{\Omega^n(V)[q,q^{-1}]}{df \wedge \Omega^{n-1}(V)[q,q^{-1}]}$$

where the differential d is taken with respect to $u \in V := (\mathbb{C}^*)^{n-1}$, see proposition 3.3.1.

Lemma 6.2.1 *The quotient $G_0/\theta G_0$ is a free $\mathbb{C}[q,q^{-1}]$-module of rank $n-1$ and*

$$([\omega_0], [u_1\omega_0], \cdots, [u_1 \cdots u_{n-2}\omega_0])$$

is a basis of it.

Proof. Let us show that the system alluded to gives a system of generators. Notice first the relations

$$u_i \frac{\partial f}{\partial u_i} = u_1 - \frac{(u_{n-1}+q)^2}{u_1 \cdots u_{n-1}} \tag{45}$$

for $i = 1, \cdots, n-2$ and

$$u_{n-1}\frac{\partial f}{\partial u_{n-1}} = \frac{u_{n-1}^2 - q^2}{u_1 \cdots u_{n-1}} \tag{46}$$

We thus have

$$u_1\frac{\partial f}{\partial u_1} - u_{n-1}\frac{\partial f}{\partial u_{n-1}} = u_1 - 2\frac{u_{n-1}+q}{u_1 \cdots u_{n-2}}$$

from which we get (equalities hold now modulo the Jacobian ideal $(\partial_{u_1} f, \cdots, \partial_{u_n} f)$)

$$u_{n-1} + q = \frac{1}{2}u_1^2 u_2 \cdots u_{n-2} \tag{47}$$

Putting this in (45), we get

$$u_{n-1} = \frac{1}{4}u_1^2 u_2 \cdots u_{n-2}$$

and, using (47),

$$u_{n-1} = q \text{ et } u_1^2 u_2 \cdots u_{n-2} = 4q$$

We deduce from this that we have indeed a system of generators because

$$u_1 = u_2 = \cdots = u_{n-2}$$

(always modulo the Jacobian ideal). This gives in particular the relations

$$u_{n-1}\omega_0 = q\omega_0 \text{ et } u_{n-2}\omega_0 = \cdots = u_1\omega_0 \tag{48}$$

in $G_0/\theta G_0$. Last, corollary 5.1.4 shows that there are no non trivial relations between the sections: for $i = 1, \cdots, n-2$, the classes of $u_1 \cdots u_i \omega_0$ and $f^i \omega_0$ are indeed proportional in $G_0/\theta G_0$. □

Let us define

- H (*resp.* H^{log}) the sub-$\mathbb{C}[\theta,\theta^{-1},q,q^{-1}]$-module (*resp.* sub-$\mathbb{C}[\theta,\theta^{-1},q]$-module) of G generated by $\varepsilon = (\varepsilon_0, \cdots, \varepsilon_{n-1})$ where ε is defined by formula (44),

- H_0 (*resp.* H_0^{log}) the sub-$\mathbb{C}[\theta, q, q^{-1}]$-module (*resp.* sub-$\mathbb{C}[\theta, q]$-module) of G generated by $(\varepsilon_0, \cdots, \varepsilon_{n-1})$,
- H_2 (*resp.* H_2^{log}) the sub-$\mathbb{C}[\theta, q, q^{-1}]$-module (*resp.* sub-$\mathbb{C}[\theta, q]$-module) of G generated by $(\varepsilon_0, \cdots, \varepsilon_{n-2})$.

We shall see that these modules are free. H_0 is the counterpart of the Brieskorn lattice G_0 in the tame case and H^{log} provides a canonical logarithmic extension of H along $q = 0$ (the eigenvalues of the residue matrix are all equal to 0). Of course, it remains to give a geometric meaning of H_0.

Proposition 6.2.2 *The matrix of ∇ takes the form, in the system of generators ε of H_0^{log},*

$$(\frac{A_0(q)}{\theta} + A_1)\frac{d\theta}{\theta} - (n-1)^{-1}\frac{A_0(q)}{\theta}\frac{dq}{q} \tag{49}$$

where

$$A_0(q) = (n-1)\begin{pmatrix} 0 & 0 & . & 0 & 2q & 0 \\ 1 & 0 & . & 0 & 0 & 2q \\ 0 & 1 & . & 0 & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & 1 & 0 & 0 \\ 0 & 0 & . & 0 & 1 & 0 \end{pmatrix}$$

and $A_1 = \mathrm{diag}(0, 1, \cdots, n-1)$.

Proof. First, we have

- $[q\frac{\partial f}{\partial q}\omega_0] = [u_1\omega_0]$,
- $[q\frac{\partial f}{\partial q}u_1 \cdots u_{n-i}\omega_0] = [u_1 \cdots u_{n-i+1}\omega_0]$ pour $i = 3, \cdots, n-1$,
- $[q\frac{\partial f}{\partial q}u_1 \cdots u_{n-2}\omega_0] = 2q[\omega_0] + 2[u_{n-1}\omega_0]$,
- $[q\frac{\partial f}{\partial q}u_{n-1}\omega_0] = q[u_1\omega_0]$

and this follows respectively from the following formulas:

- $q\frac{\partial f}{\partial q} = u_1 - u_1\frac{\partial f}{\partial u_1} + u_{n-1}\frac{\partial f}{\partial u_{n-1}}$,
- $q\frac{\partial f}{\partial q}u_1 \cdots u_{n-i} = u_1 \cdots u_{n-i+1} - u_1 \cdots u_{n-i}u_{n-2}\frac{\partial f}{\partial u_{n-2}} + u_1 \cdots u_{n-i}u_{n-1}\frac{\partial f}{\partial u_{n-1}}$ si $i = 3, \cdots, n-1$,
- $q\frac{\partial f}{\partial q}u_1 \cdots u_{n-2} = 2q + 2u_{n-1} - 2u_1 \cdots u_{n-1}\frac{\partial f}{\partial u_{n-1}}$,
- $q\frac{\partial f}{\partial q}u_{n-1} = qu_1 - qu_1\frac{\partial f}{\partial u_1} - qu_{n-1}\frac{\partial f}{\partial u_{n-1}}$,

This gives the matrix of $\nabla_{q\partial_q}$ and the remaining assertion follows from formula (36). $\square$

Proposition 6.2.3 *The $\mathbb{C}[\theta, q]$-module H_0^{log} is free of rank n and $(\varepsilon_0, \cdots, \varepsilon_{n-1})$ is a basis of it.*

Proof. Observe the following:

- H_2 is a free $\mathbb{C}[\theta, q, q^{-1}]$-module of rank $n-1$, with basis $(\varepsilon_0, \cdots, \varepsilon_{n-2})$: $\varepsilon_0, \cdots, \varepsilon_{n-2}$ are linearly independent because their classes in $G_0/\theta G_0$ are so, see proposition 3.3.1 and lemma 6.2.1. It follows that H_2^{log} is free of rank $n-1$.
- H is, by definition, of finite type and moreover equipped with a connection by proposition 6.2.2: it is thus free over $\mathbb{C}[\theta, \theta^{-1}, q, q^{-1}]$, see [26, proposition 1.2.1]. It follows that H_0 is free over $\mathbb{C}[\theta, q, q^{-1}]$. Indeed, let $\alpha_0, \cdots, \alpha_r$ be a basis of H: for $i \in \{0, \cdots, r\}$ there exists $d_i \in \mathbb{N}$ such that $\theta^{-d_0}\alpha_0, \cdots, \theta^{-d_r}\alpha_r$ generate H_0 over $\mathbb{C}[\theta, q, q^{-1}]$ and there are no non trivial relations between these sections on $\mathbb{C}[\theta, q, q^{-1}]$. It follows that H_0^{log} is free over $\mathbb{C}[\theta, q]$.
- H_2^{log} is a free sub-module of the free module H_0^{log}: the rank of H_0^{log} is therefore greater or equal than $n-1$. The free module H_0^{log} has n generators: its rank is therefore less or equal than n. It follows that the rank of H_0^{log} is equal to $n-1$ or n.
- Assume for the moment that the rank of H_0^{log} is equal to $n-1$: one would have a relation

  $$a_0(\theta, q)\varepsilon_0 + \cdots + a_{n-1}(\theta, q)\varepsilon_{n-1} = 0 \tag{50}$$

  where the $a_i(\theta, q)$'s are homogeneous polynomials in (θ, q) (recall that q is of degree $n-1$ and θ is of degree 1, see section 5.2). One would have $a_{n-1}(0, q) = 1$ because $[\varepsilon_{n-1}] = q[\varepsilon_0]$ modulo θ by equation (48), and thus $a_{n-1}(\theta, q) = 1$ by homogeneity. Because ε_{n-1} is of degree $n-1$, one would have finally

  $$\varepsilon_{n-1} = (a_0 q + b_0\theta^{n-1})\varepsilon_0 + a_1\theta^{n-2}\varepsilon_1 + \cdots + a_{n-2}\theta\varepsilon_{n-2}$$

  Apply $\theta\nabla_{q\partial_q}$ to this formula: using the computations of proposition 6.2.2, one gets

  $$\begin{aligned}(a_0 q\theta + 2a_{n-2}q\theta + a_{n-2}(a_0 q\theta + b_0\theta^n))\varepsilon_0 \\ +(a_0 q + b_0\theta^{n-1} + a_{n-2}a_1\theta^{n-1})\varepsilon_1 + (a_1\theta^{n-2} + a_{n-2}a_2\theta^{n-2})\varepsilon_2 \\ + \cdots + (a_{n-3}\theta^2 + a_{n-2}a_{n-2}\theta^2)\varepsilon_{n-2} = 2q\varepsilon_1\end{aligned}$$

  It follows that

  - $a_{n-2}b_0 = 0$
  - $a_0 + 2a_{n-2} + a_{n-2}a_0 = 0$
  - $a_0 = 2$
  - $b_0 + a_{n-2}a_1 = 0$
  - $a_i + a_{n-2}a_{i+1} = 0$ pour $i = 1, \cdots, n-3$

  The first three equalities give $a_0 = 2$, $a_{n-2} = -\frac{1}{2}$ and $b_0 = 0$. From the following ones we get $a_1 = \cdots = a_{n-3} = 0$ and finally $a_{n-2} = 0$: this is a contradiction. We conclude that the rank of H_0^{log} is not equal to $n-1$.

To sum up, H_0^{log} is free of rank n and because $(\varepsilon_0, \cdots, \varepsilon_{n-1})$ is a system of n generators it is also a basis of it. □

6.2.3 Proof of theorem 1.0.1

We keep the notations of section 6.2.2 (we refer to section 7.2.1 for a description of the small quantum cohomology of hypersurfaces that we consider here).

Theorem 6.2.4 *We have a direct sum decomposition*

$$G = H \oplus H^{\circ} \tag{51}$$

of free $\mathbb{C}[\theta, \theta^{-1}, q, q^{-1}]$*-modules where* H *is free of rank* n *and is equipped with a connection making it isomorphic to the differential system associated with the small quantum cohomology of quadrics in* $\mathbb{P}^n$*.*

Proof. The module G/H is of finite type and therefore free because it is equipped with a connection as it follows from proposition 6.2.2. We thus have the direct sum decomposition

$$G = H \oplus r(G/H)$$

where r is a section of the projection $p : G \to G/H$. This gives (51) with $H^{\circ} := r(G/H)$. The assertion about quantum cohomology follows from example 7.2.2 and formula (49) *via* the correspondence $\varepsilon_i \leftrightarrow b_i$ where b denotes the hyperplane class and b_i the i-fold cup-product of b by itself. □

It follows that the rank of G is greater or equal than n and that it is equal to n if and only if[8] $H^{\circ} = 0$. This is what happens for instance for $n = 3$ et $n = 4$, see example 5.3.1 and this is what it is expected in general, see conjecture 5.3.4.

Remark 6.2.5 *H has only two slopes,* 0 *and* 1*. In particular, H is the (localized) Fourier transform of a regular holonomic module M whose singular points run through* $C(f) \cup \{0\}$*. Moreover,*

$$\mathrm{Rank}\, H = \mathrm{Irr}(H) + \mathrm{Reg}(H)$$

where $\mathrm{Irr}(H) = n - 1$ *and* $\mathrm{Reg}(H) = d - 1$*: indeed,* $Q(\omega_0) = 0$ *where*

$$Q = \theta^n (\nabla_{\theta\partial_\theta})^n - 2q(n-1)^{n-1} n\theta(\nabla_{\theta\partial_\theta}) + 2q(n-1)^n \theta$$

and ω_0 *is cyclic.*

Remark 6.2.6 *(Metric) In order to get a whole quantum differential system it remains to construct a flat "metric" on H, see f.i [10]. If S is a ∇-flat, non degenerate bilinear form on H_0, then*

$$\begin{cases} S(\varepsilon_i, \varepsilon_j) = S(\varepsilon_0, \varepsilon_{n-1}) \in \mathbb{C}^* \theta^{n-1} \ \textit{si}\ i + j = n - 1 \\ S(\varepsilon_i, \varepsilon_j) = 0 \ \textit{otherwise} \end{cases}$$

Conversely, all flat metrics are of this kind: as A_0 is cyclic, one can argue as in [12].

[8] Notice that we do not assert in the theorem that H° is equipped with a connection.

7 Appendix: small quantum cohomology of hypersurfaces in projective spaces (overview)

We briefly recall here the definition of the small quantum cohomology of smooth hypersurfaces in projectives spaces alluded to in this paper. Our references are [3], [15], [19] and [27].

7.1 Small quantum cohomology

Given a Fano projective manifold M and a homology class $A \in H_2(M;\mathbb{Z})$ one defines Gromov-Witten invariants (three points, genus 0) $GW_A : H^*(M;\mathbb{C})^3 \to \mathbb{C}$ which satisfy the following properties:

Linearity. GW_A is linear in each variable.

Effectivity. GW_A is zero if $\int_A \omega_M < 0$, ω_M denoting the symplectic form on M.

Degree. Let x, y and z be homogeneous cohomology classes. Then $GW_A(x,y,z) = 0$ if

$$\deg x + \deg y + \deg z \neq 2\dim_{\mathbb{C}} M + 2 < c_1(M), A >$$

$c_1(M)$ denoting the first Chern class of M and $< x, A >= \int_A x$.

Initialisation. $GW_0(x,y,z) = \int_M x \cup y \cup z$.

Divisor axiom. If z is a degree 2 cohomology class one has $GW_A(x,y,z) =< z, A > GW_A(x,y,1)$.

Assume that the rank of $H^2(M;\mathbb{Z})$ is 1 and let p be a generator of it. Let $b_0, \cdots, b_s$ be a basis of $H^*(M;\mathbb{C})$ and $b^0, \cdots, b^s$ its Poincaré dual. The small quantum product $\circ_{tp}$ (for short $\circ$) is defined as follows:

$$x \circ_{tp} y = \sum_{i=0}^{s} \sum_{A \in H_2(M;\mathbb{Z})} GW_A(x,y,b_i) q^A b^i \tag{52}$$

where $q^A = \exp(tA)$. It follows from the Fano condition that the sum (52) is finite, see for instance [3, Proposition 8.1.3].

7.2 Small quantum cohomology of hypersurfaces in (weighted) projective spaces

7.2.1 In projective spaces

Assume that $M = X_d^n$ is a degree $d \geq 1$ smooth hypersurface in $\mathbb{P}^n$ and let $i : X_d^n \hookrightarrow \mathbb{P}^n$ be the inclusion. Let $p \in H^2(\mathbb{P}^n;\mathbb{C})$ be the hyperplane class and $b = i^* p$. Then $c_1(X_d^n) = (n+1-d)b$. In what follows, we will assume that $n+1-d > 0$ (Fano case). We have

$$\begin{cases} H^m(X_d^n;\mathbb{C}) = H^m(\mathbb{P}^n;\mathbb{C}) \text{ si } m < n-1 \\ H^m(X_d^n;\mathbb{C}) = H^{m+2}(\mathbb{P}^n;\mathbb{C}) \text{ si } m > n-1 \end{cases} \tag{53}$$

In particular, $H^2(X_d^n;\mathbb{C}) = H^2(\mathbb{P}^n;\mathbb{C})$ if $n \geq 4$. The cohomology ring is divided in two parts:

The ambient part. This is the space $H_{amb}(X_d^n;\mathbb{C}) := \operatorname{im} i^*$, where $i^* : H^*(\mathbb{P}^n;\mathbb{C}) \to H^*(X_d^n;\mathbb{C})$. We have $H_{amb}(X_d^n;\mathbb{C}) = \oplus_{i=0}^{n-1}\mathbb{C}b_i$ where $b_i = b \cup \cdots \cup b$ (i-times) and this is a cohomology algebra of rank n.

The primitive part. This is $P(X_d^n) := \ker i_! \subset H^{n-1}(X_d^n;\mathbb{C})$, where $i_! : H^{n-1}(X_d^n;\mathbb{C}) \to H^{n+1}(\mathbb{P}^n;\mathbb{C})$ is the Gysin morphism.

The small quantum cohomology of X_d^n preserves the ambient part $H_{amb}(X_d^n;\mathbb{C})$, see [20], [3, Chapter 11]. We thus get a subring denoted by $QH_{amb}(X_d^n;\mathbb{C})$, equipped with the product $\circ$ and which describes the small quantum product of cohomology classes coming from the ambient space $\mathbb{P}^n$: using the degree property we get, for $0 \leq m \leq n-1$,

$$b \circ b_{n-1-m} = b_{n-m} + \sum_{\ell \geq 1} L_m^\ell q^\ell b_{n-m-\ell(n+1-d)} \tag{54}$$

and

$$b \circ b_{n-1} = \sum_{\ell \geq 1} L_0^\ell q^\ell b_{n-\ell(n+1-d)} \tag{55}$$

where $L_m^\ell \in \mathbb{C}$ and $q^\ell = \exp(t\ell A)$, A denoting a generator of $H_2(X_d^n;\mathbb{Z})$; the constants L_m^ℓ vanish unless $0 \leq m \leq n-(n+1-d)\ell$ and we have $\deg q = (n+1-d)$, which is positive in the Fano case. *This is this product that we consider in these notes.*

Last, let us make the link between the small quantum cohomology and the quantum differential operators defined in section 4.2. The differential system associated with X_d^n is

$$\left\{ \begin{array}{l} \theta q\partial_q \varphi_{n-1-m}(q) = \varphi_{n-m}(q) + \sum_{\ell\geq 1} L_m^\ell q^\ell \varphi_{n-m-\ell(n+1-d)}(q) \text{ pour } m = 1, \cdots, n-1 \\ \\ \theta q\partial_q \varphi_{n-1}(q) = \sum_{\ell \geq 1} L_0^\ell q^\ell \varphi_{n-\ell(n+1-d)}(q) \end{array} \right. \tag{56}$$

see formula (54) and (55). It follows from [13] that this system can be written

$$P(\theta q\partial_q, q, \theta)\varphi_0(q) = [(\theta q\partial_q)^n - qd^d(\theta q\partial_q + \frac{1}{d}\theta)\cdots(\theta q\partial_q + \frac{d-1}{d}\theta)]\varphi_0(q) = 0 \tag{57}$$

In other words, the matrix of system (56) is conjugated to a companion matrix whose characteristic polynomial is $P(X, q, \theta)$. This allows to compute the constants L_m^ℓ.

Remark 7.2.1 *A first consequence is the formula*

$$b^{\circ n} = qd^d b^{\circ d-1} \tag{58}$$

see for instance [3, page 364], which reads $P(\theta q\partial_q, q, 0) = 0$ via the correspondences

$$b\circ \leftrightarrow \theta q\partial_q \text{ and } 1 \leftrightarrow \varphi_0 \tag{59}$$

A justification is the following: in the basis $(\varphi_0, \theta q\partial_q\varphi_0, \cdots, (\theta q\partial_q)^{n-1}\varphi_0)$ the matrix of $\theta q\partial_q$ is $\Omega_0 + \theta[\cdots]$ where Ω_0 is a matrix with coefficients in $\mathbb{C}[q]$ and whose characteristic polynomial is $P(\theta q\partial_q, q, 0)$. Up to conjugacy, the matrix Ω_0 is also the one of $\theta q\partial_q$ in the basis $(\varphi_0, \cdots, \varphi_{n-1})$.

Example 7.2.2 *Let us consider the quadric in* $\mathbb{P}^n$. *In the basis* $(\varphi_0, \cdots, \varphi_{n-1})$, *the matrix of* $\theta q \partial_q$ *takes the form*

$$\begin{pmatrix} 0 & \cdots & 2q & 0 \\ 1 & \cdots & 0 & 2q \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

It is also the matrix of $b\circ$, *using the correspondences (59).*

7.2.2 In weighted projective spaces

For smooth hypersurfaces in weighted projective spaces our references are [16], [15], [7] and [17]. Let $M = X_d^w$ be a degree $d \geq 1$ hypersurface in $\mathbb{P}(w) := \mathbb{P}(w_0, \cdots, w_n)$, satisfying the assumptions of theorem 4.1.3. Let $i : X_d^w \hookrightarrow \mathbb{P}(w)$ be the inclusion, $p \in H^2(\mathbb{P}(w); \mathbb{C})$ the hyperplane class and $b = i^* p$. By proposition 4.1.2, the first Chern class $c_1(X_d^w)$ is $(w-d)b$ and we will assume in what follows that $w - d > 0$ (Fano case, recall that $w = w_0 + \cdots + w_n$). The cohomology $H^m(\mathbb{P}(w); \mathbb{C})$ groups of the untwisted sector are of rank 1 if m is even, they vanish otherwise and

$$\begin{cases} H^m(X_d^w; \mathbb{C}) = H^m(\mathbb{P}(w); \mathbb{C}) \text{ si } m < n-1 \\ H^m(X_d^w; \mathbb{C}) = H^{m+2}(\mathbb{P}(w); \mathbb{C}) \text{ si } m > n-1 \end{cases} \tag{60}$$

see [7, Corollary 2.3.6 et 4.2.2] and [17, Theorem 7.2]. As before, we divide the cohomology ring $H^*(M; \mathbb{C})$ into an ambient part $H_{amb}(X_d^w; \mathbb{C}) := \operatorname{im} i^*$, where $i^* : H^*(\mathbb{P}(w); \mathbb{C}) \to H^*(X_d^w; \mathbb{C})$ and a primitive part. We thus have $H_{amb}(X_d^w; \mathbb{C}) = \oplus_{i=0}^{n-1} \mathbb{C} b_i$ where $b_i = b \cup \cdots \cup b$ (i-times). The small quantum product of X_d^w should preserves this ambient part and one would at the end get a subring $QH_{amb}(X_d^w; \mathbb{C})$, equipped with a product $\circ$. The differential system associated with this small quantum product looks like (compare with (56))

$$\begin{cases} \theta q \partial_q \varphi_{n-1-m}(q) = \varphi_{n-m}(q) + \sum_{\ell \geq 1} L_m^\ell q^\ell \varphi_{n-m-\ell(w-d)}(q) \text{ pour } m = 1, \cdots, n-1 \\ \\ \theta q \partial_q \varphi_{n-1}(q) = \sum_{\ell \geq 1} L_0^\ell q^\ell \varphi_{n-\ell(w-d)}(q) \end{cases} \tag{61}$$

where q is now of degree $w - d > 0$. Following [13], [16] and [15, section 5] this systems should be equivalent to the equation $P_H(\varphi_0(q)) = 0$ where P_H is the differential operator defined by formula (20). Again, one can derive from this the constants L_m^ℓ in terms of combinatorial data. A consequence is the formula

$$b^{\circ n} = q \frac{d^d}{\prod_{i=1}^n w_i^{w_i}} b^{\circ d+n-w} \tag{62}$$

as in remark 7.2.1.

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